

Large deviation asymptotics for the left tail of the sum of dependent positive random variables*

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Abstract

We study the left tail behavior of the logarithm of the distribution function of a sum of dependent positive random variables. Asymptotics are computed under the assumption that the marginal distribution functions decay slowly at zero, meaning that their logarithms are slowly varying functions. This includes parametric families such as log-normal, gamma, Weibull and many distributions from the financial mathematics literature. We show that the logarithmic asymptotics of the sum in question depend on a characteristic of the copula of the random variables which we term *weak lower tail dependence function*, and which is computed explicitly for several families of copulas in this paper. In applications, our results may be used to quantify the diversification of long-only portfolios of financial assets with respect to extreme losses. As an illustration, we compute the left tail asymptotics for a portfolio of options in the multi-dimensional Black-Scholes model.

Key words: tail behavior, large deviations, regular variation, tail dependence, portfolio diversification, Gaussian copula

1 Introduction

We consider the tail behavior of the sum of n dependent positive random variables:

$$X = \sum_{i=1}^n X_i$$

This problem has received considerable attention in the literature, but mainly in the insurance context, where the random variables X_1, \dots, X_n represent losses from individual claims, and one is interested in the *right tail* asymptotics of X , so as to estimate the probability of having a very large aggregate loss. In this setting, provided the variables X_1, \dots, X_n are sufficiently fat-tailed (subexponential),

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under various assumptions on the dependence structure, it can be shown that the right tail behavior of X is determined by the single variable with the fattest tail. We refer to [1, 2, 8, 10, 16, 17, 25, 5] and the references therein for precise statements and proofs in various contexts of this result, known as the “principle of single big jump”.

In this paper, we focus on the finance context, where the random variables X_1, \dots, X_n represent the prices of individual assets and X represents a long-only portfolio of an investor. In this context, to estimate the probability of a very large loss, one needs to focus on the *left tail* asymptotics of X . Owing to the positivity of the variables X_1, \dots, X_n , the behavior of the left tail of X turns out to be very different from that of the right tail. Indeed, for $\{X \geq x\}$ it is enough that *at least one* of X_i satisfies $X_i \geq x$, while for $X \leq x$, it is necessary that *all* X_i satisfy $X_i \leq x$. It is then intuitively clear that the dependence among X_1, \dots, X_n plays a more important role in the left-tail asymptotics than in the right-tail one.

When the variables X_1, \dots, X_n are independent, the tail behavior of X can be studied with characteristic function / Laplace transform methods. For example, the following result, which covers, e.g., the gamma distribution, is a straightforward consequence of the Tauberian theorem (see [3]).

Proposition 1. *Assume that X_1, \dots, X_n are independent and that for each i , the distribution function F_i of X_i satisfies*

$$F_i(x) \sim \frac{x^{\rho_i} l_i(x)}{\Gamma(1 + \rho_i)}, \quad x \rightarrow 0,$$

where $\rho_i \geq 0$ and l_i is slowly varying at zero. Then, the distribution function F of X satisfies

$$F(x) \sim \frac{\prod_{i=1}^n \Gamma(1 + \rho_i)}{\Gamma(1 + \rho_1 + \dots + \rho_n)} \prod_{i=1}^n F_i(x), \quad x \rightarrow 0.$$

However, for distribution functions which are not regularly varying, the product of marginal probabilities $\mathbb{P}[X_i \leq x]$ does not provide a good approximation for the tail of X . For instance, when X_i follows the inverse Gaussian law with density

$$f_i(x) = \frac{\mu_i}{x^{\frac{3}{2}} \sqrt{2\pi}} e^{-\frac{(\lambda x - \mu_i)^2}{2x}},$$

the sum X has density

$$f_i(x) = \frac{\sum_{i=1}^n \mu_i}{x^{\frac{3}{2}} \sqrt{2\pi}} e^{-\frac{(\lambda x - \sum_{i=1}^n \mu_i)^2}{2x}}.$$

As $x \rightarrow 0$, the distribution functions can be shown to satisfy

$$F_i(x) \sim \frac{2x}{\mu \sqrt{2\pi}} e^{-\frac{\mu^2}{2x} + \lambda \mu} \quad \text{and} \quad F(x) \sim \frac{2x}{\sqrt{2\pi} \sum_{i=1}^n \mu_i} e^{-\frac{(\sum_{i=1}^n \mu_i)^2}{2x} + \lambda \sum_{i=1}^n \mu_i},$$

which means that $F(x)$ decays much faster than $\prod_{i=1}^n F_i(x)$ as $x \rightarrow 0$.

When the variables X_1, \dots, X_n are dependent, the law of X is more difficult to analyze, and very few results are available in the literature. Wüthrich [24] considers the left-tail asymptotics for a sum of identically distributed random variables in the domain of attraction of Weibull and Gumbel distributions (for the minimum), with dependence given by an Archimedean copula with a regularly varying generator. He finds that in these cases

$$\mathbb{P}[X \leq nx] \sim C\mathbb{P}[X_1 \leq x]$$

for some constant C , as u tends to the lower bound of the support of distribution of X_1 . In other words, the tail dependence in the Archimedean copula is so strong that asymptotically all variables become perfectly correlated and no diversification effects come into play.

However, for weaker tail dependence patterns, the situation may be very different. For instance, when $X_i, i = 1, \dots, n$ are exponentials of components of a Gaussian vector (in other words, log-normal random variables with a Gaussian copula), the tail behavior of X may depend on the entire covariance matrix of the Gaussian vector, and the left tail of X may be much thinner than the tails of X_1, \dots, X_n . This has been shown in [9] for $n = 2$ and more recently in [11] in the general case. For example, when X_1, \dots, X_n are identically distributed such that $\log X_i \sim N(\mu, \sigma^2)$, and the correlation between X_i and X_j is equal to ρ for all $i \neq j$ with $|\rho| < 1$,

$$\mathbb{P}[X \leq x] \sim C \left(\log \frac{1}{x} \right)^{-\frac{1+n}{2}} \exp \left(-\frac{n}{2\sigma^2(1+\rho(n-1))} \left\{ \log \frac{x}{n} - \mu \right\}^2 \right),$$

for some constant C . We see that for any value of ρ the tail of X is thinner than the tail of X_1 and for $\rho = 0$, $F(x)$ decays much faster than $\prod_{i=1}^n F_i(x)$ as $x \rightarrow 0$.

These motivating examples show that it does not seem possible, under sufficiently general assumptions, to express the asymptotics of $F(x)$ in terms of the asymptotics of $F_i(x)$ for $i = 1, \dots, n$. For this reason, in this paper we consider a weaker logarithmic formulation, and study the limiting behavior of

$$\frac{\log \mathbb{P}[X_1 + \dots + X_n \leq x]}{\min_i \log \mathbb{P}[X_i \leq x]} \tag{1}$$

as $x \rightarrow 0$.

From the mathematical point of view, the limit of this expression, which we compute explicitly in many cases, provides a large deviations estimate for the left tail of the sum of positive dependent random variables in terms of their marginal distributions.

From the applied point of view, it can be seen as a measure of asymptotic diversification of a portfolio of dependent risks. A value close to 1 indicates that the portfolio is poorly diversified, since its behavior under extreme scenarios is similar to that of the component with the thinnest tail. By contrast,

a large value corresponds to good diversification. Portfolio diversification with respect to extreme risks has recently been studied in the context of fat-tailed distributions satisfying the property of multivariate regular variation [21, 20, 6]. The present paper complements these references by studying the left tail of a portfolio of positive assets, to which the multivariate regular variation theory does not readily apply.

We compute the limit of (1) under the following assumptions on the marginal laws.

- The logarithms of distribution functions of X_i are slowly varying at 0. This assumption includes all distributions with regularly varying left tail as well as parametric families such as log-normal, gamma, Weibull and many distributions from the financial mathematics literature.
- The logarithms of the distribution functions of X_i are equivalent, up to a constant, to a common function:

$$\log F_i(x) \sim \lambda_i \log F(x).$$

This assumption ensures that the laws of components have similar asymptotic behavior, but nevertheless is not very restrictive: for example, X_i with different i -s can follow log-normal distributions with different parameters, or have regularly varying tails with different indices.

Under the above assumptions, we show that the limit of (1) can be expressed in terms of the coefficients λ_i and of a characteristic of the copula of X_1, \dots, X_n , which we term *weak lower tail dependence function*. This function is related to the *weak lower tail dependence coefficient* introduced in the literature in the two-dimensional case (see remark 1) and is defined by

$$\chi(\alpha_1, \dots, \alpha_n) = \lim_{u \rightarrow 0} \frac{\min_i \log u^{\alpha_i}}{\log C(u^{\alpha_1}, \dots, u^{\alpha_n})}, \quad \alpha_1, \dots, \alpha_n \geq 0.$$

In the particular case when the logarithmic tails of X_1, \dots, X_n are all equivalent to each other (e.g., when $\lambda_1 = \dots = \lambda_n$), it follows that the limit of (1) does not depend on the marginal distribution of X_1, \dots, X_n and is determined exclusively by the copula-dependent quantity

$$\chi = \lim_{u \rightarrow 0} \frac{\log u}{\log C(u, \dots, u)}.$$

Note that for each fixed x , the value (1) depends both on the copula and the marginal distributions. This result also provides a new interpretation of the weak tail dependence coefficient and shows that for analyzing the tail behavior of sums of dependent random variables (portfolios of dependent risks), this measure of tail dependence is more relevant than the strong lower tail dependence coefficient defined by

$$\lambda = \lim_{u \rightarrow 0} \frac{C(u, \dots, u)}{u}.$$

Our second main contribution is to compute the weak tail dependence function for commonly used families of copulas. Of particular interest is the result for the Gaussian copula since in applications one often has to deal with random variables X_i which are not Gaussian or log-normal themselves, but whose dependence is nevertheless given by the Gaussian copula. For example, this happens when X_1, \dots, X_d are non-linear functions of risk factors which form a Gaussian random vector (see section 4 for a concrete example). More generally, the Gaussian copula is by far the most popular way of introducing dependence between non-Gaussian risk factors in finance and other domains [15, 19]. It is therefore important to understand the tail behavior of $\sum_{i=1}^d X_i$ when the dependence of X_1, \dots, X_d is given by a Gaussian copula, for various marginal distributions. We show that even though the strong tail dependence coefficient is zero for the Gaussian copula (see e.g. [7]), the weak tail dependence function has a nontrivial form and allows to quantify in a precise fashion the diversification effect of Gaussian dependence.

Remarks on notation Throughout this paper, we write $f \sim g$ as $x \rightarrow a$ whenever $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1$ and $f \lesssim g$ whenever $\limsup_{x \rightarrow a} \frac{f(x)}{g(x)} \leq 1$. We recall that a function f is called slowly varying as $x \rightarrow 0$ whenever $\lim_{x \rightarrow 0} \frac{f(\alpha x)}{f(x)} = 1$ for all $\alpha > 0$.

We also recall that the copula of a random vector (Y_1, \dots, Y_d) is a function $C : [0, 1]^d : [0, 1]$, satisfying the assumptions

- dC is a positive measure in the sense of Lebesgue-Stieltjes integration,
- $C(u_1, \dots, u_d) = 0$ whenever $u_k = 0$ for at least one k ,
- $C(u_1, \dots, u_d) = u_k$ whenever $u_u = 1$ for all $i \neq k$,

and such that

$$\mathbb{P}[Y_1 \leq y_1, \dots, Y_d \leq y_d] = C(\mathbb{P}[Y_1 \leq y_1], \dots, \mathbb{P}[Y_d \leq y_d]), \quad (y_1, \dots, y_d) \in \mathbb{R}^d.$$

A copula exists by Sklar's theorem and is uniquely defined whenever the marginal distributions of Y_1, \dots, Y_d are continuous. We refer to [22] for details on copulas.

2 Tail asymptotics

Definition 1. The *weak lower tail dependence function* $\chi(\alpha_1, \dots, \alpha_n)$ of a copula C is defined by

$$\chi(\alpha_1, \dots, \alpha_n) = \lim_{u \rightarrow 0} \frac{\min_i \log u^{\alpha_i}}{\log C(u^{\alpha_1}, \dots, u^{\alpha_n})},$$

whenever the limit exists and is finite for all $\alpha_1, \dots, \alpha_n \geq 0$ such that $\alpha_k > 0$ for at least one k . The *weak lower tail dependence coefficient* of a copula C is defined by

$$\chi = \chi(1, \dots, 1) = \lim_{u \rightarrow 0} \frac{\log u}{\log C(u, \dots, u)},$$

whenever the limit exists.

Remark 1. The weak lower tail dependence coefficient defined above is closely related to the “coefficient of tail dependence” introduced in [18] in the 2-dimensional setting. It was further studied in [4] under the name “dependence measure $\bar{\chi}$ ” and in a number of other papers including [23]. In particular, [13] gives the values of this index (in the two-dimensional case) for various families of copulas. The copula appears in the denominator to obtain an expression which is increasing with respect to the concordance order of copulas, meaning that stronger dependence corresponds to larger values of χ .

The weak lower tail dependence function $\chi(\alpha_1, \dots, \alpha_n)$ of a copula is order 0 homogeneous: for all $r > 0$,

$$\chi(r\alpha_1, \dots, r\alpha_n) = \chi(\alpha_1, \dots, \alpha_n).$$

It is increasing with respect to the concordance order of copulas and admits the following bounds (the upper bound is due to the Frechet-Hoeffding upper bound on the copula):

$$0 \leq \chi(\alpha_1, \dots, \alpha_n) \leq 1.$$

The upper bound is attained for the complete dependence copula $C_{\parallel}(u_1, \dots, u_n) = \min(u_1, \dots, u_n)$. On the other hand, for the independence copula $C_{\perp}(u_1, \dots, u_n) = u_1 \dots u_n$, we get

$$\chi(\alpha_1, \dots, \alpha_n) = \frac{\max_i \alpha_i}{\sum_i \alpha_i}.$$

Weak lower tail dependence functions for various copula families will be computed in section 3

The following theorem is the main result of this paper.

Theorem 1. *Let X_1, \dots, X_n be random variables with values in $(0, \infty)$ with marginal distribution functions F_1, \dots, F_n and copula C satisfying the following assumptions.*

- *For each $k = 1, \dots, n$, F_k is slowly varying at zero and satisfies*

$$\log F_k(x) \sim \lambda_k \log F(x)$$

for some constant $\lambda_k > 0$ and some function F .

- *The copula C admits a weak lower tail dependence function χ .*

Then,

$$\lim_{x \downarrow 0} \frac{\log \mathbb{P}[X_1 + \dots + X_n \leq x]}{\min_i \log \mathbb{P}[X_i \leq x]} = \frac{1}{\chi(\lambda_1, \dots, \lambda_n)}.$$

Proof. We first establish an upper bound on $\mathbb{P}[X_1 + \dots + X_n \leq x]$.

$$\mathbb{P}[X_1 + \dots + X_n \leq x] \leq \mathbb{P}[X_1 \leq x, \dots, X_n \leq x] = C(F_1(x), \dots, F_n(x)).$$

By assumption of the theorem, for any $\varepsilon > 0$ and x small enough,

$$F_k(x) \leq F(x)^{\lambda_k(1-\varepsilon)}, \quad k = 1, \dots, n.$$

Therefore,

$$\mathbb{P}[X_1 + \dots + X_n \leq x] \leq C(F(x)^{\lambda_1(1-\varepsilon)}, \dots, F(x)^{\lambda_n(1-\varepsilon)})$$

and by definition of the weak lower tail dependence function, for x small enough, we then have

$$\mathbb{P}[X_1 + \dots + X_n \leq x] \leq F(x)^{\chi^{-1}(\lambda_1, \dots, \lambda_n)(1-\varepsilon)^2 \max_i \lambda_i}.$$

On the other hand,

$$\mathbb{P}[X_1 + \dots + X_n \leq x] \geq \mathbb{P}[X_1 \leq \frac{x}{n}, \dots, X_n \leq \frac{x}{n}],$$

which, by a computation similar to the above one leads to the lower bound

$$P[X_1 + \dots + X_n \leq x] \geq F(x/n)^{\chi^{-1}(\lambda_1, \dots, \lambda_n)(1+\varepsilon)^2 \max_i \lambda_i}.$$

Taking the logarithms and using the fact that ε is arbitrary and $\log F$ is slowly varying shows that

$$\lim_{x \downarrow 0} \frac{\log \mathbb{P}[X_1 + \dots + X_n \leq x]}{\max_i \lambda_i \log F(x)} = \chi^{-1}(\lambda_1, \dots, \lambda_n)$$

and therefore

$$\lim_{x \downarrow 0} \frac{\log \mathbb{P}[X_1 + \dots + X_n \leq x]}{\log \min_i \mathbb{P}[X_i \leq x]} = \chi^{-1}(\lambda_1, \dots, \lambda_n).$$

□

Corollary 1. *Let X_1, \dots, X_n be random variables with values in $(0, \infty)$ with marginal distribution functions F_1, \dots, F_n and copula C satisfying the following assumptions.*

- *For each $k = 1, \dots, n$, F_k is slowly varying at zero and satisfies*

$$\log F_k(x) \sim \log F(x)$$

for some function F .

- *The copula C admits a weak lower tail dependence coefficient χ .*

Then,

$$\lim_{x \downarrow 0} \frac{\log \mathbb{P}[X_1 + \dots + X_n \leq x]}{\min_i \log \mathbb{P}[X_i \leq x]} = \frac{1}{\chi}.$$

3 Weak lower tail dependence function for common copula families

The degree of tail dependence in a given copula function may be quantified by the *strong tail dependence coefficient*, which is defined (for the left tail) by

$$\lambda_L = \lim_{u \downarrow 0} \frac{C(u, \dots, u)}{u},$$

whenever the limit exists. When $\lambda > 0$, the copula is said to have strong tail dependence in the left tail. Strong tail dependence coefficients for different copula families are listed, for instance, in [22, 13]. In particular, it is known that the Gaussian copula does not have strong tail dependence. The following simple result shows that if a copula has strong tail dependence in the left tail, its weak tail dependence function is equal to the upper bound. Weak and strong tail dependence are thus “orthogonal” notions, which are relevant for different dependence regimes.

Proposition 2. *Assume that a copula function C has strong tail dependence in the left tail with coefficient $\lambda_L > 0$. Then, the weak lower tail dependence function of C is equal to the upper bound:*

$$\chi(\alpha_1, \dots, \alpha_n) = 1.$$

Proof. From the definition of λ_L , for any $\varepsilon > 0$ and u sufficiently small,

$$C(u, \dots, u) \geq (\lambda_L - \varepsilon)u.$$

Using the fact that the copula is increasing in each argument, we have, for u sufficiently small,

$$\frac{\log C(u^{\alpha_1}, \dots, u^{\alpha_n})}{\log u} \leq \frac{\log(\lambda_L - \varepsilon) + \max(\alpha_1, \dots, \alpha_n) \log u}{\log u},$$

which shows that

$$\limsup_{u \downarrow 0} \frac{\log C(u^{\alpha_1}, \dots, u^{\alpha_n})}{\log u} = \max(\alpha_1, \dots, \alpha_n).$$

Combining this with the Frechet-Hoeffding upper bound on the copula, the proof is complete. \square

The above proposition implies in particular that for all copulas of elliptical distributions with regularly varying tails, including, in particular, the t -copula, which are known to have strong tail dependence [14], the weak tail dependence function is equal to 1.

The Gaussian copula with correlation matrix R is the unique copula of any Gaussian vector with correlation matrix R and nonconstant components (it does not depend on the mean vector and on the variances of the components). The

following proposition characterizes the weak lower tail dependence function of the Gaussian copula. In this proposition, we define

$$\Delta_n := \{w \in \mathbb{R}^n : w_i \geq 0, i = 1, \dots, n, \sum_{i=1}^n w_i = 1\}.$$

Proposition 3. *Let C be a Gaussian copula with correlation matrix R with $\det R \neq 0$. Then,*

$$\chi(\alpha_1, \dots, \alpha_n) = \max_i \alpha_i \min_{w \in \Delta_n} w^T \Sigma w, \quad \text{for all } \alpha_1, \dots, \alpha_n > 0,$$

where the matrix Σ has coefficients $\Sigma_{ij} = \frac{R_{ij}}{\sqrt{\alpha_i \alpha_j}}$, $1 \leq i, j \leq n$.

Proof. Let (X_1, \dots, X_n) be a centered Gaussian vector with covariance matrix Σ defined above. From results in [12], one can deduce that there exist positive constants c and C such that, for all z sufficiently small,

$$\frac{c}{|z|^{\bar{n}}} e^{-\frac{z^2}{2 \inf_{w \in \Delta_n} w^T \Sigma w}} \leq \mathbb{P}[X_1 \leq z, \dots, X_n \leq z] \leq \frac{C}{|z|^{\bar{n}}} e^{-\frac{z^2}{2 \inf_{w \in \Delta_n} w^T \Sigma w}}$$

where $\bar{n} = \#\{i = 1, \dots, n : \bar{w}_i > 0\}$ and $\bar{w} = \arg \inf_{w \in \Delta_n} w^T \Sigma w$. This means that

$$\log \mathbb{P}[X_1 \leq z, \dots, X_n \leq z] \sim -\frac{z^2}{2 \inf_{w \in \Delta_n} w^T \Sigma w}$$

as $z \rightarrow -\infty$. Applying this to a single Gaussian variable yields $\mathbb{P}[X_i \leq z] \sim -\frac{z^2 \alpha_i}{2}$ as $z \rightarrow -\infty$. Now combine these estimates to get, for ε and z small enough,

$$\begin{aligned} -\frac{z^2(1+\varepsilon)}{2 \inf_{w \in \Delta_n} w^T \Sigma w} &\leq \log \mathbb{P}[X_1 \leq z, \dots, X_n \leq z] = \log C(\mathbb{P}[X_1 \leq z], \dots, \mathbb{P}[X_n \leq z]) \\ &\leq \log C(e^{-\frac{z^2 \alpha_1(1-\varepsilon)}{2}}, \dots, e^{-\frac{z^2 \alpha_n(1-\varepsilon)}{2}}). \end{aligned}$$

Letting $u = e^{-\frac{z^2(1-\varepsilon)}{2}}$, this leads to

$$\frac{1+\varepsilon}{(1-\varepsilon) \inf_{w \in \Delta_n} w^T \Sigma w} \log u \leq \log C(u^{\alpha_1}, \dots, u^{\alpha_n}).$$

Dividing by $\min_i \log u^\alpha$, and using the fact that ε is arbitrary, we finally get

$$\max_i \alpha_i \inf_{w \in \Delta_n} w^T \Sigma w \geq \limsup_{u \rightarrow 0} \frac{\min_i \log u^\alpha}{\log C(u^{\alpha_1}, \dots, u^{\alpha_n})}.$$

The lower bound may be obtained in a similar fashion. \square

Finally we recall that given a function $\phi : [0, 1] \rightarrow [0, \infty]$ which is continuous, strictly decreasing and such that its inverse ϕ^{-1} is completely monotonic, the Archimedean copula with generator ϕ is defined by

$$C(u_1, \dots, u_n) = \phi^{-1}(\phi(u_1) + \dots + \phi(u_n)).$$

The following simple result gives the weak lower tail dependence function for an Archimedean copula. The case when $\log \phi^{-1}$ is regularly varying includes for example the Gumbel copula with $\phi^{-1}(t) = \exp(-t^{1/\theta})$ and several other families.

Proposition 4. *Let C be an Archimedean copula with generator function ϕ .*

(i). *If $\log \phi^{-1}$ is regularly varying at $+\infty$ with index $\lambda > 0$, then,*

$$\chi(\alpha_1, \dots, \alpha_n) = \frac{\max(\alpha_1, \dots, \alpha_n)}{(\alpha_1^{1/\lambda} + \dots + \alpha_n^{1/\lambda})^\lambda}$$

(ii). *If $\log \phi^{-1}$ is slowly varying at $+\infty$, then*

$$\chi(\alpha_1, \dots, \alpha_n) = 1$$

Proof. Assume first that $\log \phi^{-1}$ is regularly varying with index $\lambda > 0$. By definition of χ ,

$$\begin{aligned} \chi(\alpha_1, \dots, \alpha_n) &= \lim_{u \rightarrow 0} \frac{\max(\alpha_1, \dots, \alpha_n) \log u}{\log \phi^{-1}(\phi(u^{\alpha_1}) + \dots + \phi(u^{\alpha_n}))} \\ &= \frac{\max(\alpha_1, \dots, \alpha_n) \log u}{\log \phi^{-1}(\phi(u^{\alpha_1}) + \dots + \phi(u^{\alpha_n}))} \\ &= \lim_{u \rightarrow 0} \frac{\max(\alpha_1, \dots, \alpha_n) \log \phi^{-1}(\phi(u))}{\log \phi^{-1}(\phi(e^{\alpha_1 \log u}) + \dots + \phi(e^{\alpha_n \log u}))} \end{aligned}$$

By the inversion theorem for regularly varying functions [3], the function $u \mapsto \phi(e^u)$ is regularly varying at $-\infty$ with index $\frac{1}{\lambda}$. Therefore, for any $\varepsilon > 0$ and u sufficiently small,

$$\begin{aligned} (1 - \varepsilon)(\alpha_1^{1/\lambda} + \dots + \alpha_n^{1/\lambda})\phi(u) &\leq \phi(e^{\alpha_1 \log u}) + \dots + \phi(e^{\alpha_n \log u}) \\ &\leq (1 + \varepsilon)(\alpha_1^{1/\lambda} + \dots + \alpha_n^{1/\lambda})\phi(u), \end{aligned}$$

and we conclude using the regular variation of $\log \phi^{-1}$ and the fact that ε is arbitrary. The proof for the case when $\log \phi^{-1}$ is slowly varying is similar. \square

4 An application to finance

In this section we show how the asymptotic results obtained in this note may be used to analyze the tail behavior of a portfolio of options. Fix a time horizon T and let (X_1, \dots, X_n) denote the vector of logarithmic returns of n risky assets under real-world measure over this time horizon. The asset prices at date T are then given by $S_i = e^{X_i}$ for $i = 1, \dots, n$ where we have assumed without loss of generality that the initial values of all assets are normalized to 1. We suppose that the n risky assets follow the multidimensional Black-Scholes model. This

means that the distribution of the vector (X_1, \dots, X_n) is Gaussian, and we denote by ΣT its covariance matrix and by μT its mean vector.

We are interested in the tail behavior a long-only portfolio of European call options written on n risky assets. To simplify the discussion we assume that the portfolio contains exactly one option on each of the risky assets, but the setting can obviously be extended to an arbitrary number of options. The log-strikes of the options will be denoted by (k_1, \dots, k_n) and the maturity dates by (T_1, \dots, T_n) , where $T_i > T$ for $i = 1, \dots, n$. Assuming that the interest rate is zero, the price of i -th option at date T is given by the Black-Scholes formula:

$$P_i = e^{X_i} N(d_1) - e^{k_i} N(d_2), \quad d_{1,2} = \frac{X_i - k_i}{\sigma_i \sqrt{T_i - T}} \pm \frac{\sigma_i \sqrt{T_i - T}}{2}, \quad \sigma_i = \sqrt{\Sigma_{ii}},$$

where N is the standard normal distribution function.

The following proposition clarifies the asymptotic behavior of the probability $\mathbb{P}[P_1 + \dots + P_n \leq z]$ as $z \rightarrow 0$.

Proposition 5. *As $z \rightarrow 0$,*

$$\log \mathbb{P}[P_1 + \dots + P_n \leq z] \sim -\frac{\log \frac{1}{z}}{\inf_{w \in \Delta_n} w^\perp R w},$$

where R is a $n \times n$ matrix with elements given by $R_{ij} = \frac{\Sigma_{ij} T}{\sigma_i \sigma_j \sqrt{(T_i - T)(T_j - T)}}$.

Proof. P_1, \dots, P_n are obviously increasing and continuous functions of the Gaussian random variables (X_1, \dots, X_n) . Therefore, the copula of (P_1, \dots, P_n) is the Gaussian copula with correlation matrix with elements $\rho_{ij} = \frac{\Sigma_{ij}}{\sigma_i \sigma_j}$. It remains to characterize the asymptotic behavior of the distribution functions of P_1, \dots, P_n .

Let $\tilde{X}_i = \frac{X_i - \mu_i T}{\sigma_i \sqrt{T}}$ for $i = 1, \dots, n$ and define

$$f_i(x) = e^{\mu_i T + x \sigma_i \sqrt{T}} N(d_1(x)) - e^{k_i} N(d_2(x)),$$

$$d_{1,2}(x) = x \sqrt{\frac{T}{T - T_i}} - \frac{\mu_i T + k_i}{\sigma_i \sqrt{T_i - T}} \pm \frac{\sigma_i \sqrt{T_i - T}}{2}.$$

Then, \tilde{X}_i is a standard normal random variable. From the well-known equivalence

$$N(x) \sim \frac{e^{-\frac{x^2}{2}}}{|x| \sqrt{2\pi}}, \quad x \rightarrow -\infty,$$

one easily deduces that

$$f_i(x) \sim \frac{\sigma_i (T_i - T)^{\frac{3}{2}}}{x^2 T \sqrt{2\pi}} e^{k_i - \frac{d_2^2(x)}{2}}, \quad x \rightarrow -\infty.$$

Taking the logarithm, we obtain

$$\log f_i(x) \sim -\frac{x^2 T}{2(T_i - T)}, \quad x \rightarrow -\infty$$

and

$$f_i^{-1}(u) \sim \sqrt{2 \frac{T_i - T}{T} \log \frac{1}{u}}, \quad u \rightarrow 0.$$

Therefore, the distribution function of P_i satisfies

$$\log \mathbb{P}[P_i \leq x] = \log N(f_i^{-1}(x)) \sim -\frac{f_i^{-1}(x)^2}{2} \sim \frac{T_i - T}{T} \log \frac{1}{x}, \quad x \downarrow 0,$$

so that the assumptions of Theorem 1 are satisfied with $\lambda_i = \frac{T_i - T}{T}$ and $F(x) = \frac{1}{x}$ and the result follows by applying Proposition 3 and Theorem 1. \square

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